

Discrete Quantum Gravity: II. Simplicial complexes, irreps of $SL(2,C)$, and a Lorentz invariant weight in a state sum model.

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Abstract. In part I of [1] we have developed the tensor and spin representation of $SO(4)$ in order to apply it to the simplicial decomposition of the Barrett-Crane model. We attach to each face of a triangle the spherical function constructed from the Dolginov-Biedenharn function.

In part II we apply the same technique to the Lorentz invariant state sum model. We need three new ingredients: the classification of the edges and the corresponding subspaces that arises in the simplicial decomposition, the irreps of $SL(2,C)$ and its isomorphism to the bivectors appearing in the 4-simplices, the need of a zonal spherical function from the intertwining condition of the tensor product for the simple representations attached to the faces of the simplicial decomposition.

Keywords: $SL(2,C)$ group, unitary representation, simplicial decomposition, spherical functions, intertwining condition in the tensor product.

1. Introduction.

Part II deals mainly with the representation theory for relativistic spin networks in quantum gravity. In section 2 we give some elementary properties of vector and bivector analysis in Minkowski space $M(1,3)$. In section 3 we start from general representation theory, valid also for the Euclidean version of Part I. Then we describe the global unitary irreps of the group $SL(2,C)$, the universal covering group of $SO(1,3,R)$. Section 4 deals in general with the notions of intertwiners and invariants, spherical harmonics and spherical functions. In sections 5, 6, 7 we review the irreps of $SL(2,C)$ and obtain the spherical functions occurring in the state sum for relativistic spin networks.

2. Vectors, bivectors, subspaces, and simplices.

To deal with relativistic spin networks we must go from the Euclidean space R^4 discussed in part I to Minkowski space $M(1,3)$. We summarize



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in this section some properties of vectors and bivectors and use results given in [3] pp. 56-76.

1 Def: A vector $x \in R^n$ is time-like if $(x \circ x) > 0$, space-like if $(x \circ x) < 0$, light-like if $(x \circ x) = 0$. A time-like vector is called positive (negative) if $x_1 > 0$ ($x_1 < 0$). We define the vector norm by $\|x\| := \sqrt{(x \circ x)}$.

2 Def: A simple bivector $b = x \wedge y$ has components $(x \wedge y)_{\mu\nu} := x_\mu y_\nu - x_\nu y_\mu$ and squared norm

$$\langle b, b \rangle := 2((x \circ x)(y \circ y) - (x \circ y)^2). \quad (1)$$

3 Def: [3] p.61: Let V be a vector subspace of R^n .

- (1) V is time-like if and only if it has a time-like vector,
- (2) V is space-like if and only if every nonzero vector of V is space-like,
- (3) V is light-like otherwise.

4 Def:

- (1) b is space-like bivector if $\langle b, b \rangle > 0$,
- (2) b is time-like bivector if $\langle b, b \rangle < 0$,
- (3) b is a simple bivector if $\langle b, {}^* b \rangle = 0$.

Classification of pairs of vectors by scalar products, span and simple bivectors according to [3]:

5.1 Prop: Let both x, y be positive (negative) time-like. They span a time-like subspace $V = \{x, y\}$. Then by [3] p.62 eq. 3.1.6

$$\begin{aligned} x \circ y &= \|x\| \|y\| \cosh(x, y), \\ \langle b, b \rangle &= 2\|x\|^2 \|y\|^2 (1 - (\cosh(x, y))^2) < 0. \end{aligned} \quad (2)$$

5.2 Prop: Let x, y be space-like and span a space-like subspace $V = \{x, y\}$. Then by [3] p.71 eq. 3.2.6

$$\begin{aligned} x \circ y &= \|x\| \|y\| \cos(x, y), \\ \langle b, b \rangle &= 2\|x\|^2 \|y\|^2 (1 - (\cos(x, y))^2) > 0. \end{aligned} \quad (3)$$

5.3 Prop: Let x, y be space-like and span a time-like subspace $V = \{x, y\}$. Then by [3] p.73 eq. 3.2.7

$$\begin{aligned} x \circ y &= \|x\| \|y\| \cosh(x, y), \\ \langle b, b \rangle &= 2\|x\|^2 \|y\|^2 (1 - (\cosh(x, y))^2) < 0. \end{aligned} \quad (4)$$

5.4 Prop: Let x be space-like and y be time-like positive. Then $V = \{x, y\}$ is time-like and by [3] p. 75 eq. 3.2.8

$$\begin{aligned} x \circ y &= \|x\| \|y\| \sinh(x, y), \\ \langle b, b \rangle &= 2\|x\|^2 \|y\|^2 (1 - (\sinh(x, y))^2) = \text{indefinite}. \end{aligned} \quad (5)$$

6 Def: An m -simplex S^m with a vertex at 0 can be defined as the convex hull of the points from a set of m linearly independent edge vectors x^1, x^2, \dots, x^m as $S^m := \text{conv}(\lambda_1 x^1, \lambda_2 x^2, \dots, \lambda_m x^m), 0 \leq \lambda_j \leq 1$. Then we can generalize Prop 1.2 to

7 Prop: Let the m -simplex $S^m, m < n$ be the convex hull of m space-like linearly independent edge vectors x^1, x^2, \dots, x^m which span a space-like subspace V . Then all subsimplices $S^p, p < m$ of S^m have space-like edge vectors which span space-like subspaces.

Proof. All the edge vectors of the simplex S^m are in the space-like subspace spanned by x^1, x^2, \dots, x^m . Any subspace of a space-like subspace is space-like. We can prove the existence of these simplices by choosing in Minkowski space a set of up to $n - 1$ orthogonal space-like unit vectors. Certainly these span a space-like subspace.

For the case $m = n$ we can still choose n space-like linearly independent vectors. But as these provide a basis for the full Minkowski space, they must include time-like vectors and so cannot span a space-like subspace.

8 Prop: Classification of simple bivectors $b = a_1 \wedge a_2$ attached to a triangle by their scalar product:

If a_1, a_2 span a space-like subspace then from **Prop 5.2**: $\langle b, b \rangle > 0$.

If a_1, a_2 span a time-like subspace then from **Prop 5.1 and 5.3**: $\langle b, b \rangle < 0$.

9 Prop: The second implication cannot be reversed. Only if we exclude simple bivectors formed from a space-like and a time-like vector then the squared norm of the bivector determines the type of spanned subspace.

We see from these results that, for a simplicial triangulation in Minkowski space $M(1, 3)$, care must be taken already with the choice of triangular faces and bivectors. We are not aware of considerations of this point in the literature on relativistic spin networks.

3. Irreducible unitary representations of $SL(2, C)$.

When we switch from Euclidean space R^4 to Minkowski space $M(1, 3)$, we must deal with the unitary irreps of the group $SL(2, C)$ which is the universal covering group of the group $SO(1, 3, R)$. We start in subsection 3.1 with some general properties of representations and then describe in subsections 3.2 and 3.3 the unitary irreps of $SL(2, C)$.

3.1. OPERATORS, AUTOMORPHISMS AND IRREDUCIBLE REPRESENTATIONS.

Consider in a Hilbert space of complex-valued functions of a complex variable with scalar product

$$\langle \phi, \psi \rangle = \int \overline{\phi(z)} \psi(z) d\mu(z), \quad d\mu(z) := dRe(z)dIm(z) \quad (6)$$

a set $\{T\}$ of linear operators T with inverses T^{-1} . Denote complex conjugation of functions by overlining.

10 Def: Define the transposed, the conjugate and the adjoint operator respectively by

$$\begin{aligned} T^T &: \langle \phi, T^T \psi \rangle = \langle \psi, T\phi \rangle, \\ T^C &: \overline{\langle \phi, T\psi \rangle} = \langle \phi', T^C \psi' \rangle, \\ T^\dagger &: \langle \phi, T^\dagger \psi \rangle = \langle T\phi, \psi \rangle. \end{aligned} \quad (7)$$

The states ψ', ϕ' are defined in eq. 23 below.

The following involutive operator automorphisms $\Sigma : T \rightarrow \Sigma(T)$ obeying $\Sigma(T_\alpha T_\beta) = \Sigma(T_\alpha)\Sigma(T_\beta)$ are called [2] p.139 contragredient, conjugate and contragredient-conjugate:

$$\begin{aligned} \Sigma_1 &: T \rightarrow (T^T)^{-1} = (T^{-1})^T, \\ \Sigma_2 &: T \rightarrow T^C, \\ \Sigma_3 &: T \rightarrow (T^\dagger)^{-1} = (T^{-1})^\dagger. \end{aligned} \quad (8)$$

Next consider a representation of a group G by the set $\{T\}$ of operators,

$$g \in G \rightarrow T_g, \quad T_e = I, \quad T_{g^{-1}} = (T_g)^{-1}, \quad T_{g_1 g_2} = T_{g_1} T_{g_2}. \quad (9)$$

If the representation is unitary we find

$$(T_g^\dagger)^{-1} = T_g, \quad T_g^C = (T_g^{-1})^T, \quad \Sigma_3(T) = T, \quad \Sigma_1(T) = \Sigma_2(T). \quad (10)$$

Denote by $g \in G \rightarrow T_g^\lambda$ a unitary irreducible representation with irrep label λ . By the operator automorphisms eq. 8 and by unitarity eq. 10 it follows that for any irrep λ there is a conjugate irrep $(T_g^\lambda)^C$ whose irrep label we denote by λ^C .

3.2. PRINCIPAL SERIES OF IRREDUCIBLE REPRESENTATIONS OF $SL(2, C)$.

Consider now a group of complex-valued matrices $g \in G$. With respect to this group we have in analogy to eq.8 three involutive group automorphisms obeying $\sigma : \sigma(g_1 g_2) = \sigma(g_1)\sigma(g_2)$,

$$\begin{aligned}\sigma_1 : \quad g &\rightarrow (g^{-1})^T, \\ \sigma_2 : \quad g &\rightarrow \bar{g}, \\ \sigma_3 : \quad g &\rightarrow (g^{-1})^\dagger.\end{aligned}\tag{11}$$

The three group automorphisms eq. 11 allow to construct from a fixed representation $g \rightarrow T_g$ three new representations $T_{\sigma_1(g)}$, $T_{\sigma_2(g)}$, $T_{\sigma_3(g)}$.

These representations should be carefully distinguished from the ones defined by eq. 8, since the automorphisms of operators eq. 7 in representation space in general cannot be pulled down to automorphisms eq. 11 of group elements, as will be seen in eq. 29 below!

We define the complex valued polynomials

$$p(z, \bar{z}) = \sum C_{\alpha\beta} z^\alpha \bar{z}^\beta$$

as the basic states of the spinor representations:

$$T_a p(z, \bar{z}) = (\beta z + \delta)^k (\bar{\beta} \bar{z} + \bar{\delta})^n p\left(\frac{\alpha z + \gamma}{\beta z + \delta}, \frac{\bar{\alpha} \bar{z} + \bar{\gamma}}{\bar{\beta} \bar{z} + \bar{\delta}}\right)$$

for any

$$a \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, C), \alpha\delta - \beta\gamma = 1$$

This representation with labels $(l_0, l_1) = \left(\frac{k-n}{2}, \frac{k+n}{2} + 1\right)$, $k, n \in N$, is irreducible and finite dimensional, see [1] eq. (20). If we enlarge this representation with complex values, k, n we get:

$$T_a f(z, \bar{z}) = (\beta z + \delta)^{l_0 + l_1 - 1} (\bar{\beta} \bar{z} + \bar{\delta})^{l_0 - l_1 - 1} f\left(\frac{\alpha z + \gamma}{\beta z + \delta}, \frac{\bar{\alpha} \bar{z} + \bar{\gamma}}{\bar{\beta} \bar{z} + \bar{\delta}}\right)$$

the representation becomes infinite dimensional. For l_0 integer or half integer and l_1 arbitrary complex, the representation is irreducible.

We follow [2] p. 565 and define the principal series of irreps of $SL(2, C)$ on a Hilbert space of functions in a complex variable z with scalar product eq. 6.

11 Def: For the group element

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad g^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \in SL(2, C), \quad \alpha\delta - \beta\gamma = 1, \tag{12}$$

we adopt the operator

$$(T_g^{[m,\rho]} \psi)(z) := |\beta z + \delta|^{m+i\rho-2} (\beta z + \delta)^{-m} \psi \left(\frac{\alpha z + \gamma}{\beta z + \delta} \right). \quad (13)$$

The representation is characterized by the integer/real numbers $[m, \rho]$. These numbers determine the eigenvalues of two Casimir operators C_1, C_2 , [4] p.167 according to

$$C_1 \psi^{[m,\rho]} = -\frac{(m^2 - \rho^2 - 4)}{2} \psi^{[m,\rho]}, C_2 \psi^{[m,\rho]} = m\rho \psi^{[m,\rho]}. \quad (14)$$

For the inverse element we find

$$(T_{g^{-1}}^{[m,\rho]} \psi)(z) = |-\beta z + \alpha|^{m+i\rho-2} (-\beta z + \alpha)^{-m} \psi \left(\frac{\delta z - \gamma}{-\beta z + \alpha} \right) \quad (15)$$

To find the adjoint operator we substitute in the integral expression for the left-hand side of

$$\langle \phi, T_g^{[m,\rho]} \psi \rangle = \langle (T_g^{[m,\rho]})^\dagger \phi, \psi \rangle \quad (16)$$

the new variable

$$z' = \frac{\alpha z + \gamma}{\beta z + \delta}. \quad (17)$$

To transform the measure $d\mu(z)$ we compute

$$\begin{aligned} \frac{\partial z'}{\partial z} &= 1/(\beta z + \delta)^2, \\ \frac{\partial \bar{z}'}{\partial \bar{z}} &= 1/\overline{(\beta z + \delta)^2}, \\ \frac{d\mu(z')}{d\mu(z)} &= 1/|\beta z + \delta|^4 = |-\beta z' + \alpha|^4. \end{aligned} \quad (18)$$

After some rewriting we obtain for the adjoint operator

$$(T_g^{[m,\rho],\dagger} \phi)(z') = |-\beta z' + \alpha|^{m+i\rho-2} (-\beta z' + \alpha)^{-m} \phi \left(\frac{\delta z' - \gamma}{-\beta z' + \alpha} \right). \quad (19)$$

Comparing with the operator $T_{g^{-1}}$ (15) we verify the unitarity eq. 10 of the representation eq. 13,

$$(T_g^{[m,\rho]})^\dagger = T_{g^{-1}}^{[m,\rho]}. \quad (20)$$

From unitarity it follows by eq. 10, that, besides of the original irrep eq. 13, we need to consider only the conjugate irrep. To determine the

operation of conjugation it is better to consider the elements ψ, ϕ as functions of z, \bar{z} . Then eq. 13 becomes

$$(T_g^{[m,\rho]} \psi)(z, \bar{z}) := |\beta z + \delta|^{m+i\rho-2} (\beta z + \delta)^{-m} \psi\left(\frac{\alpha z + \gamma}{\beta z + \delta}, \frac{\overline{\alpha z} + \overline{\gamma}}{\overline{\beta z} + \overline{\delta}}\right). \quad (21)$$

The complex conjugation of this equation is

$$\overline{(T_g^{[m,\rho]} \psi)(z, \bar{z})} := |\overline{\beta z} + \overline{\delta}|^{m-i\rho-2} (\overline{\beta z} + \overline{\delta})^{-m} \overline{\psi}\left(\frac{\alpha z + \gamma}{\beta z + \delta}, \frac{\overline{\alpha z} + \overline{\gamma}}{\overline{\beta z} + \overline{\delta}}\right). \quad (22)$$

We wish to rewrite this expression as the action of a conjugate operator on a complex-valued function. To this purpose we define

$$\overline{\psi(z, \bar{z})} =: \psi'(z', \bar{z}'), \quad \overline{\phi(z, \bar{z})} =: \phi'(z', \bar{z}'), \quad z' = \bar{z}, \bar{z}' = z. \quad (23)$$

These definitions allow to rewrite eq. 22 as

$$\begin{aligned} & \overline{(T_g^{[m,\rho]} \psi)(z, \bar{z})} \\ &= |\overline{\beta z} + \overline{\delta}|^{m-i\rho-2} (\overline{\beta z} + \overline{\delta})^{-m} \psi'\left(\frac{\overline{\alpha z'} + \overline{\gamma}}{\overline{\beta z'} + \overline{\delta}}, \frac{\alpha \bar{z}' + \gamma}{\beta \bar{z}' + \delta}\right). \end{aligned} \quad (24)$$

Comparing with the definition eq. 21 we determine the conjugate operator as

$$\overline{(T_g^{[m,\rho]} \psi)(z, \bar{z})} := (T_g^{[m,\rho],C} \psi')(z', \bar{z}') = (T_{\sigma_2(g)}^{[m,-\rho]} \psi')(z', \bar{z}'), \quad (25)$$

and obtain for the conjugate irrep $[m, \rho]^C = [m, -\rho]$. Moreover with eq. 10 one easily verifies

$$T_g^{[m,\rho],C} =: (T_g^{[m,\rho]})^C = (T_{g^{-1}}^{[m,\rho]})^T = T_{\sigma_2(g)}^{[m,-\rho]}. \quad (26)$$

We keep the group automorphism σ_2 from eq. 11 since we wish to consider the expressions in eq. 26 as homomorphisms $g \rightarrow T_g$.

12 Prop: The irreps labelled by $[m, \rho]$ eq. 13 of the principal series of $SL(2, C)$ are unitary. The conjugate irreps are labelled by $[m, -\rho]$ and given by eq. 25.

Finally we turn to the group automorphisms eq. 11. For the group $SL(2, C)$ we have the particular result

$$(g^{-1})^T = q g q^{-1}, \quad q = -q^{-1} = \bar{q} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (27)$$

Since $q \in SL(2, C)$, σ_1 becomes an inner automorphism. eq. 11 implies

$$\sigma_1(g) = q g q^{-1}, \quad \sigma_3(g) = \bar{q} \bar{g} \overline{q^{-1}} = q \bar{g} q^{-1} = q \sigma_2(g) q^{-1} \quad (28)$$

The automorphisms σ_2, σ_3 are related by conjugation in the group. In terms of the irreducible representations, we find from these expressions

$$\begin{aligned} T_{(g^{-1})^T}^{[m,\rho]} &= T_q^{[m,\rho]} T_g^{[m,\rho]} T_{q^{-1}}^{[m,\rho]} \sim T_g^{[m,\rho]}, \\ T_{\overline{g}}^{[m,\rho]} &= (T_g^{[m,-\rho]})^C, \\ T_{(g^{-1})^\dagger}^{[m,\rho]} &= T_q^{[m,\rho]} T_{\overline{g}}^{[m,\rho]} T_{q^{-1}}^{[m,\rho]} \sim T_{\overline{g}}^{[m,\rho]}. \end{aligned} \quad (29)$$

3.3. COMPLEMENTARY SERIES OF IRREDUCIBLE REPRESENTATIONS OF $SL(2,C)$.

We consider the Hilbert space of complex-valued functions of a complex variable with scalar product

$$(f_1, f_2) = \iint |z_1 - z_2|^{-2+\sigma} f_1(z_1) \overline{f_2(z_2)} dz_1 dz_2 \quad (30)$$

where the transposed, conjugate and adjoint operators are defined as before in eq. 7, and similarly the contragradient, the conjugate and the contragradient conjugate involutive automorphisms are defined as in eq. 8.

Following [2] page 573, we define the complementary series of irreps of $SL(2,C)$ on a Hilbert space with scalar product eq. 30.

13 Def: For the group element

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, g^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \in SL(2, C), \alpha\delta - \beta\gamma = 1, \quad (31)$$

we adopt the operator expression

$$(T_g^{[0,\sigma]} \psi)(z) = |\beta z + \delta|^{-2-\sigma} \psi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) \quad (32)$$

and the inverse operator

$$(T_{g^{-1}}^{[0,\sigma]} \psi)(z) = |-\beta z + \alpha|^{-2-\sigma} \psi\left(\frac{\delta z - \gamma}{-\beta z + \alpha}\right), \quad (33)$$

where σ is a real number $0 < \sigma < 2$.

The eigenvalues of the two Casimir operators are

$$C_1 \psi^{[0,\sigma]} = -\frac{\sigma^2 - 4}{2} \psi^{[0,\sigma]}, \quad C_2 \psi^{[0,\sigma]} = 0. \quad (34)$$

To find the adjoint operator, we substitute in the integral expression eq. 7 with the help of eq. 30 the new variables

$$z'_1 = \frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}, \quad z'_2 = \frac{\alpha z_2 + \gamma}{\beta z_2 + \delta} \quad (35)$$

with the transformed measure

$$dz_1 = \frac{dz'_1}{(-\beta z'_1 + \alpha)^4}, \quad dz_2 = \frac{dz'_2}{(-\beta z'_2 + \alpha)^4}. \quad (36)$$

After simplification we obtain

$$(T_g^{[0,\sigma]\dagger} \psi)(z') = |-\beta z' + \alpha|^{-2-\sigma} \psi \left(\frac{\delta z' - \gamma}{-\beta z' + \alpha} \right). \quad (37)$$

Comparing with eq. 33 we verify the unitarity of the representation eq. 32

$$T_g^{[0,\sigma]\dagger} = T_{g^{-1}}^{[0,\sigma]} \quad (38)$$

To determine the operation of conjugation we follow the same steps eq. 21 to eq. 26 for the principal series, keeping in mind that $m = 0, \rho = i\sigma$ is a pure imaginary number. We obtain

$$T_g^{[0,\sigma]C} = T_{\sigma_2(g)}^{[0,\sigma]}, \quad (39)$$

where σ_2 is defined in eq. 11.

4. Right action invariants from two irreps.

In this section we use representation theory to examine the notions of intertwining and invariants as they appear in relation with discrete quantum gravity.

We switch for convenience to a bracket notation. Our results hold true for general unitary irreps of a group and in particular for the irreps of $SO(4, R)$ analyzed in part I. Consider for the unitary irrep $\lambda = [m, \rho]$ and two group elements g_1, g_2 the matrix element

$$\langle \lambda\mu | T_{g_1 g_2}^{[m,\rho]} | \lambda\mu' \rangle = \sum_{\nu} \langle \lambda\mu | T_{g_1}^{[m,\rho]} | \lambda\nu \rangle \langle \lambda\nu | T_{g_2^{-1}}^{[m,\rho]} | \lambda\mu' \rangle. \quad (40)$$

By construction this matrix element is invariant under the right action

$$g_1 \rightarrow g_1 g, \quad g_2 \rightarrow g_2 g. \quad (41)$$

We transform the matrix elements for the group element g_2 by

$$\begin{aligned} \langle \lambda\nu | T_{g_2^{-1}}^{\lambda} | \lambda\mu' \rangle &= \langle \lambda\nu | (T_{g_2}^{\lambda})^{\dagger} | \lambda\mu' \rangle \\ &= \overline{\langle \lambda\mu' | T_{g_2}^{\lambda} | \lambda\nu \rangle} = \langle \lambda^C \mu'^C | T_{\sigma_2(g_2)}^{\lambda^C} | \lambda^C \nu^C \rangle. \end{aligned} \quad (42)$$

In the last line we introduced the conjugate irrep and the superscripts C for its row and column labels. Substituting eq. 42 into eq. 40 and using eq. 26 we get

$$\begin{aligned} & \langle \lambda\mu | T_{g_1 g_2^{-1}}^\lambda | \lambda\mu' \rangle \\ &= \sum_\nu \langle \lambda\mu | T_{g_1}^\lambda | \lambda\nu \rangle \langle \lambda^C \mu'^C | T_{\sigma_2(g_2)}^{\lambda^C} | \lambda^C \nu^C \rangle. \end{aligned} \quad (43)$$

Next we wish to pass from the invariant eq. 40 to the intertwining or Kronecker coupling of two conjugate irreps. Consider a right-hand Kronecker coupling of two irreps λ, λ' for two different group elements g_1, g_2 , and its decomposition under the right actions eq. 41,

$$\begin{aligned} & \sum_{\nu, \nu'} \langle \lambda\mu | T_{g_1 g}^\lambda | \lambda\nu \rangle \langle \lambda'\mu' | T_{g_2 g}^{\lambda'} | \lambda'\nu' \rangle \langle \lambda\nu, \lambda'\nu' | \lambda^*\mu^* \rangle \\ &= \sum_{\sigma, \sigma'} \sum_{\nu, \nu'} \langle \lambda\mu | T_{g_1}^\lambda | \lambda\sigma \rangle \langle \lambda'\mu' | T_{g_2}^{\lambda'} | \lambda'\sigma' \rangle \langle \lambda\sigma | T_g^\lambda | \lambda\nu \rangle \\ & \quad \langle \lambda'\sigma' | T_g^{\lambda'} | \lambda'\nu' \rangle \langle \lambda\nu, \lambda'\nu' | \lambda^*\mu^* \rangle. \end{aligned} \quad (44)$$

In this expression the sum over Wigner coefficients and functions of g only yields

$$\begin{aligned} & \sum_{\nu, \nu'} \langle \lambda\sigma | T_g^\lambda | \lambda\nu \rangle \langle \lambda'\sigma' | T_g^{\lambda'} | \lambda'\nu' \rangle \langle \lambda\nu, \lambda'\nu' | \lambda^*\mu^* \rangle \\ &= \sum_{\mu^{*'}} \langle \lambda\sigma, \lambda'\sigma' | \lambda^*\mu^{*'} \rangle \langle \lambda^*\mu^{*'} | T_g^{\lambda^*} | \lambda^*\mu^* \rangle. \end{aligned} \quad (45)$$

The only term on the right-hand side of this expression independent of g is the one with $(\lambda^*, \mu^*) = (\lambda^*, \mu^{*'}) = (0, 0)$ whose Wigner coefficients couple to the identity irrep $\lambda = 0$. Therefore if we make this choice in the left hand side of the first line in eq. 44 we must get an expression invariant under the substitution eq. 41,

$$\sum_{\nu, \nu'} \langle \lambda\mu | T_{g_1}^\lambda | \lambda\nu \rangle \langle \lambda'\mu' | T_{g_2}^{\lambda'} | \lambda'\nu' \rangle \langle \lambda\nu, \lambda'\nu' | 00 \rangle. \quad (46)$$

By its form and invariance, this expression must coincide with eq. 40 up to a constant. Comparing these expressions we find the Wigner coefficients for eq. 46 up to normalization:

14 Prop: The Kronecker coupling to an invariant requires two conjugate representations λ, λ^C of eq. 46 and has the Wigner coefficients

$$\begin{aligned} \lambda' &= \lambda^C, \quad [m', \rho'] = [m, -\rho], \\ \langle \lambda\nu, \lambda'\nu' | 00 \rangle &= \delta(\lambda', \lambda^C) \delta(\nu', \nu^C). \end{aligned} \quad (47)$$

The simplicity of this expressions results from the introduction of the conjugate representation labels in eq. 42. For a general analysis of Wigner coefficients for $SL(2, C)$ principal series representations we refer to [6].

5. Representations of the algebra of $SL(2, \mathbb{C})$

In order to discuss relativistic spherical harmonics and zonal spherical functions we must write the irreps of $SL(2, C)$ as given in eqs. 12, 13 in a form which is explicitly reduced under the subgroup $SU(2)$. This will give the relativistic counterpart of the Gelfand-Zetlin basis used in the Euclidean case of part I.

For this purpose we expand the basis $\psi^{[m, \rho]}$ in terms of the orthogonal Hahn polynomials of imaginary argument [5] [7] with the help of complexified Clebsch-Gordan coefficients.

Given the generators $(J_1, J_2, J_3) = \vec{J}$ of $SU(2)$ and of pure Lorentz transformations $(K_1, K_2, K_3) = \vec{K}$ satisfying the commutation relations:

$$\begin{aligned} [J_p, J_q] &= i\varepsilon_{pqr} J_r & \vec{J}^+ &= \vec{J} & , & p, q, r = 1, 2, 3 \\ [J_p, K_q] &= i\varepsilon_{pqr} K_r & \vec{K}^+ &= \vec{K} & & \\ [K_p, K_q] &= -i\varepsilon_{pqr} J_r & & & & \end{aligned} \quad (48)$$

we obtain the unitary representations of the algebra of $SL(2, \mathbb{C})$ in the basis where the operators J_3 and \vec{J}^2 are diagonal, namely: $J_3\psi_{JM} = M\psi_{JM}$, $\vec{J}^2\psi_{JM} = J(J+1)\psi_{JM}$

It is possible also to construct complexified operators

$$\vec{A} = \frac{1}{2}(\vec{J} + i\vec{K}), \quad \vec{B} = \frac{1}{2}(\vec{J} - i\vec{K}), \quad \vec{A}^+ = \vec{B}$$

that leads to the commutation relations of two independent angular momenta:

$$\begin{aligned} [A_p, A_q] &= i\varepsilon_{pqr} A_r \\ [B_p, B_q] &= i\varepsilon_{pqr} B_r \\ [A_p, B_q] &= 0 \end{aligned} \quad (49)$$

Since J_3 and K_3 commute we construct for a fixed irrep $[m, \rho]$ the representations of these operators in the basis where J_3 and K_3 are diagonal [5],

$$\begin{aligned} J_3\phi_{m_1 m_2} &= M\phi_{m_1 m_2}, \quad K_3\phi_{m_1 m_2} = \lambda\phi_{m_1 m_2}, \text{ hence} \\ A_3\phi_{m_1 m_2} &= \frac{1}{2}(M + i\lambda)\phi_{m_1 m_2} \equiv m_1\phi_{m_1 m_2} \\ B_3\phi_{m_1 m_2} &= \frac{1}{2}(M - i\lambda)\phi_{m_1 m_2} \equiv m_2\phi_{m_1 m_2} \end{aligned}$$

Notice that λ is a real continuous parameter, but m_1 and m_2 are complex conjugate and $\bar{m}_1 = m_2$

For the Casimir operators we have

$$\begin{aligned} C_1 &= \left(\vec{J}^2 - \vec{K}^2 \right) \psi_{JM} = (l_0^2 + l_1^2 - 1) \psi_{JM} \\ C_2 &= \left(\vec{J} \cdot \vec{K} \right) \psi_{JM} = l_0 l_1 \psi_{JM} \end{aligned}$$

or in the $[m, \rho]$ notation

$$\begin{aligned} C_1 \psi^{[m, \rho]} &= \frac{1}{2} (m^2 - \rho^2 - 4) \psi^{[m, \rho]} \\ C_2 \psi^{[m, \rho]} &= m \rho \psi^{[m, \rho]} \end{aligned} \quad (50)$$

We shall extend the analysis to the irreps $[0, \sigma]$ of the completely degenerate series eq. 32.

6. Complexified Clebsch-Gordan coefficients and the representation of the boost operator

In order to connect the basis ψ_{JM} and $\phi_{m_1 m_2}$ we can use the complexified Clebsch-Gordan coefficients:

$$\psi_{JM} = \int_{-\infty}^{\infty} d\lambda \langle m_1 m_2 | JM \rangle \phi_{m_1 m_2} \quad (51)$$

We have used integration because λ is a continuous parameter. It can be proved that these coefficients are related to the Hahn polynomials of imaginary argument [5]

$$\langle m_1 m_2 | JM \rangle = f \frac{\sqrt{\omega(\lambda)}}{d_{J-M}} p_{J-M}^{(M-m, M+m)}(\lambda, \rho), \quad m = m_1 + m_2, \quad f \bar{f} = 1, \quad (52)$$

for the principal series,

$$\langle m_1 m_2 | JM \rangle = f \sqrt{\omega(\lambda)} d_{J-M}^{-1} q_{J-M}^{(M)}(\lambda, \sigma), \quad m = m_1 + m_2, \quad f \bar{f} = 1 \quad (53)$$

for the complementary series,

where

$$\omega(\lambda) = \frac{1}{4\pi} \left| \Gamma \left(\frac{M-m+1}{2} + \frac{i\lambda-\rho}{2} \right) \Gamma \left(\frac{m+\mu+1}{2} + i\frac{\lambda+\rho}{2} \right) \right|^2 \quad (54)$$

and

$$(d_n)^2 = \frac{\Gamma(M-m+n+1) \Gamma(M+m+n+1) |\Gamma(M+i\rho+n+1)|^2}{n! (2M+2n+1) \Gamma(2M+n+1)} \quad (55)$$

for the principal series and similar expression for the complementary series [5].

With the help of eqs. 51, 52 and 53 we can construct the representation for the boost operator, or the Biedenharn-Dolginov function, namely [5],

$$\begin{aligned} & d_{JJ'M}^{[m,\rho]}(\tau) \\ &= \int_{-\infty}^{\infty} d_{J-M}^{-1} p_{J-M}^{(M-m,M+m)}(\lambda, \rho) \exp(-i\tau\lambda) d_{J-M}^{-1} p_{J'-M}^{(M-m,M+m)}(\lambda, \rho) \omega(\lambda) d\lambda \end{aligned} \quad (56)$$

for the principal series,

$$\begin{aligned} & d_{JJ'M}^{[0,\sigma]}(\tau) \\ &= \int_{-\infty}^{\infty} d_{J-M}^{-1} q_{J-M}^{(M)}(\lambda, \sigma) \exp(-i\tau\lambda) d_{J'-M}^{-1} q_{J'-M}^{(M)}(\lambda, \sigma) \omega(\lambda) d\lambda \end{aligned}$$

for the complementary series.

7. Relativistic spherical harmonics on $SL(2, C)/SU(2)$, intertwining, and spherical functions.

We consider irreps and spherical harmonics on the homogeneous space $SL(2, C)/SU(2)$. Here $SU(2)$ is the stability group of the point $P_0 = (1, 0, 0, 0) \in M(1, 3)$, and the homogeneous space is the hyperboloid $H^3 < M(1, 3)$ which replaces the 3-sphere $S^3 < R^4$ used in part I [1]. We show that the general intertwining of pairs analyzed in section 4 from spherical functions yields zonal spherical functions.

Consider those irreps λ of $SL(2, C)$ which admits once and only once the irrep $J = 0$ of the compact subgroup $SU(2)$. An orthogonal irrep set of functions on the coset space $SL(2, C)/SU(2)$ can be taken as the set of matrix elements

$$\langle [m, \rho](JM = 00) | T_g^\lambda | [m, \rho](J'M') \rangle =: \langle [m, \rho] \downarrow (00) | T_g^\lambda | [m, \rho](J'M') \rangle, \quad (57)$$

where we introduced $\downarrow (00)$ for the subduction to the identity irreps of the subgroup $SU(2)$ to distinguish it from the notation (00) used for the identity irreps of the full group $SL(2, C)$ in eq. 46. Clearly the matrix elements in eq. 57 are invariant under the substitution $g \rightarrow ug$, $u \in SU(2)$ and so may be assigned to the left cosets $SL(2, C)/SU(2)$. One could decompose the measure on group space and from the orthogonality of the irreps obtain an orthogonality rule for these irrep functions which play the role of relativistic spherical harmonics. These relativistic spherical harmonics for $SO(1, 3, R)$ or $SL(2, C)$ must be distinguished

from the ones of the group $SO(4, R)$ discussed in part I section 5.2 and Theorem 1. Vilenkin and Klimyk in [10] vol. 2 p. 28 discuss the relativistic spherical harmonics in relation with Δ - and \square -harmonics.

In the models of discrete relativistic quantum gravity, these spherical harmonics can be assigned as fields to the 3-simplices or tetrahedra. We shall show that this and only this assignment yields spherical functions at the triangular faces of the relativistic spin network. Geometric sharing of simplicial boundaries is transcribed into the intertwining condition of the fields. For two tetrahedra sharing a triangular face it follows that the corresponding irreps must intertwine, i.e. can be coupled to the identity irrep of $SL(2, C)$. This coupling was analyzed in general in the previous subsection. We expand the labels in eq. 43 according to

$$\lambda\mu \rightarrow \lambda\mu \downarrow (00), \quad \lambda^C\mu^C \rightarrow \lambda^C\mu^C \downarrow (00), \quad (58)$$

and in addition drop the left-hand multiplicity label μ since for the irreps of the groups $G = (SO(4, R), SL(2, C))$, the identity irrep $J = 0$ of the stability group $SU(2)$ occurs once and only once.

15 Def: Zonal spherical functions for the irrep λ of $G = (SO(4, R); SL(2, C))$ with subgroup $SU(2)$, in a basis where G is explicitly reduced with respect to this subgroup, are defined by

$$f^\lambda(g) := \langle \lambda \downarrow (00) | T_g^\lambda | \lambda \downarrow (00) \rangle. \quad (59)$$

Clearly the spherical functions eq. 59 are invariant under the substitution $g \rightarrow u_1 g u_2$, $(u_1, u_2) \in SU(2)$. Therefore the zonal spherical functions eq. 59 can be assigned to the double cosets $SU(2) \backslash SL(2, C) / SU(2)$ and must be distinguished from the spherical harmonics eq. 57. The spherical functions eq. 59 are the relativistic counterparts of the ones considered in part I section 5.2.

Applying now eq. 43 we find for the intertwining of two spherical harmonics an expression in terms of a zonal spherical function eq. 59,

$$\begin{aligned} & \left[\sum_\nu \langle \lambda \downarrow (00) | T_{g_1}^\lambda | \lambda \nu \rangle \langle \lambda^C \downarrow (00) | T_{\sigma_2(g_2)}^{\lambda^C} | \lambda^C \nu^C \rangle \right] \\ &= f^\lambda(g_1 g_2^{-1}) = \langle \lambda \downarrow (00) | T_{g_1 g_2^{-1}}^\lambda | \lambda \downarrow (00) \rangle. \end{aligned} \quad (60)$$

This expression is invariant under the right action $(g_1, g_2) \rightarrow (g_1 q, g_2 q)$, $q \in SL(2, C)$.

The Wigner coefficients have been suppressed in view of eq. 47. The expression in the last line defines a zonal spherical function with respect to $SL(2, C) > SU(2)$.

16 Prop: The intertwining of two spherical harmonics on $SL(2, C)/SU(2)$ belonging to conjugate irreps λ, λ^C yields a single zonal spherical function eq. 60 of the product $g_1 g_2^{-1}$ of group elements.

We shall derive the spherical functions for $SL(2, C) > SU(2)$ in section 8.

We add four remarks to this result:

1: Conjugate irreps. Although the two intertwining irreps are λ, λ^C , the spherical function in eq. 60 belongs to the single irreps λ . This asymmetry is removed upon considering the inverse element $g_2 g_1^{-1}$ which by eq. 42 leads to a spherical function belonging to the irrep λ^C .

2: Reduction to $SU(2)$. The occurrence of a single irrep $\downarrow (00)$ of $SU(2)$ puts a constraint on the irrep $[m, \rho]$, [2] p. 567. In particular the irreps $[m, 0]$ are ruled out, see [4] p. 163.

3: Self-conjugacy for irreps of $SL(2, C)$. In the application to quantum gravity, the intertwining condition at a triangular face of a simplex in general requires two conjugate and hence inequivalent irreps. Since a simplex in $M(1, 3)$ is bounded by five tetrahedra, the pairwise intertwining of irreps for these tetrahedra implies a single self-conjugate irrep at each tetrahedron of a simplex.

In two special cases, $\lambda = [m, 0]$ or $\lambda = [0, \rho]$, the irrep and its conjugate are equivalent. This follows from eq. 47 and from the equivalence $[m, \rho] \sim [-m, -\rho]$ stated with proof in [4] p. 168, in accord with identical eigenvalues eq. 14 of the two Casimir operators. The irrep $[m, 0]$ is ruled out as explained in **2**. The restriction to the special irreps $[0, \rho], [0, \sigma]$ of $SL(2, C)$ would be the relativistic analog of the notion of simple irreps as used in quantum gravity, see section 5 of part I [1]. In [8] p. 3105 the authors argue that the complementary series irreps have Plancherel measure zero and therefore can be neglected for relativistic spin networks.

4: Self-conjugacy for irreps of $SO(4, R)$. In case of $SO(4, R)$ considered in part I [1], the irreps can be expressed by pairs of irreps of $SU(2)$. Therefore the conjugate irreps obey $\overline{D}^{(j_1, j_2)} \sim D^{(j_1, j_2)}$ and so are self-conjugate. This applies in particular to simple irreps $D^{(j_2 j_2)}$.

8. Zonal spherical function for the group $SO(1, 3, R)$.

Given the unitary representation T_g of the group $SL(2, C)$ and the identity representation $(JM) = (00)$ of the subgroup $SU(2)$, the zonal spherical function is defined as in Definition 15 eq. 59.

From the properties of the spherical function it is sufficient to take the representation of the Lorentz boost. From eq. 56 we have with

$$M = M' = 0, J = J' = 0$$

$$\begin{aligned} f^{[m,\rho]}(\tau) &= \langle \psi_{JM=00} | \exp(-iK_3\tau) | \psi_{J'M'=00} \rangle \quad (61) \\ &= \int d_0^{-2} e^{-i\lambda\tau} [p_0^{(0,0)}]^2 \omega(\lambda) d\lambda \\ &= \int_{-\infty}^{\infty} \frac{e^{-i\lambda\tau}}{|\Gamma(1+i\rho)|^2} \frac{1}{4\pi} \left| \Gamma\left(\frac{1}{2} + i\frac{\lambda+\rho}{2}\right) \Gamma\left(\frac{1}{2} + i\frac{\lambda-\rho}{2}\right) \right|^2 d\lambda \\ &= \int_{-\infty}^{\infty} e^{-i\lambda\tau} \frac{\sinh(\pi\rho)}{4\rho} \frac{1}{\cosh(\pi(\frac{\lambda+\rho}{2}))} \frac{1}{\cosh(\pi(\frac{\lambda-\rho}{2}))} d\lambda \end{aligned}$$

where we have used the properties of Γ functions. From the residue theorem at the poles $\lambda = \mp\rho + (2n+1)i$, $n = 0, 1, 2, \dots$, we easily obtain for $\tau < 0$

$$f^{[0,\rho]}(\tau) = i \frac{\exp(i\rho\tau) - \exp(-i\rho\tau)}{\rho} \exp \tau \sum_{n=0}^{\infty} (\exp(2\tau))^n = \frac{1}{\rho} \frac{\sin(\rho\tau)}{\sinh(\tau)} \quad (62)$$

for the principal series with $l_0 = 0$, $l_1 = i\rho$, see [4] p. 166. Identical expressions can be obtained for $\tau > 0$ if we apply the residue theorem to eq. 61 at the poles $\lambda = \mp\rho - (2n+1)i$, $n = 0, 1, 2, \dots$. In the application of the residue theorem, we have to check that the integrand of eq. 61 goes to zero when $|\lambda| \rightarrow \infty$ in the upper half-plane for $\tau < 0$, or in the lower half-plane for $\tau > 0$. This can be proven very easily in general and in the particular case of $\lambda = \mp\rho + (2n+1)i$, $n = 0, 1, 2, \dots$ by L'Hospital's rule.

We obtain in the same way, see [4] p. 186,

$$f^{[0,\sigma]}(\tau) = \frac{1}{\sigma} \frac{\sinh(\sigma\tau)}{\sinh(\tau)} \quad (63)$$

for the complementary series, with $l_0 = 0$, $l_1 = \sigma$, $|\sigma| < 1$.

9. A $SO(1,3,R)$ invariant for the state sum of a spin foam model

As in the case of euclidean $SO(4)$ invariant model, we take a non degenerate finite triangulation of a 4-manifold. We consider the 4-simplices in the homogeneous space $SO(1,3,R)/SO(3) \simeq H^3$, the hyperboloid $\{x | x \cdot x = 1, x^0 > 0\}$ and define the bivectors b on each face of the 4-simplex, that can be space-like, null or timelike ($\langle b, b \rangle > 0, = 0, < 0$, respectively).

In order to quantize the bivectors, we take the isomorphism $b = *L \left(b^{ab} = \frac{1}{2} \varepsilon^{abcd} L_d^e g_{ec} \right)$ with g a Minkowski metric.

The condition for b to be a simple bivector $\langle b, *b \rangle = 0$, gives $C_2 = \langle L, *L \rangle = \vec{J} \cdot \vec{K} = m\rho = 0$

We have two cases:

1) $\rho = 0$, $C_1 = \langle L, L \rangle = \vec{J}^2 - \vec{K}^2 = m^2 - 1 > 0$; L , space-like, b time-like,

2) $m = 0$, $C_1 = \vec{J}^2 - \vec{K}^2 = -\rho^2 - 1 < 0$; L , time like, b space like (remember, the Hodge operator $*$ changes the signature)

In case 2) b is space-like, $\langle b, b \rangle > 0$. Expanding this expression in terms of space-like vectors x, y ,

$$\begin{aligned} b_{\mu\nu} b^{\mu\nu} &= (x_\mu y_\nu - x_\nu y_\mu)(x^\mu y^\nu - x^\nu y^\mu) = \\ &= \|x\|^2 \|y\|^2 - \|x\|^2 \|y\|^2 \cos^2 \eta(x, y) = \|x\|^2 \|y\|^2 \sin^2 \eta(x, y) \end{aligned} \quad (64)$$

where $\eta(x, y)$ is the Lorentzian space-like distance between x and y ; this result gives a geometric interpretation between the parameter ρ and the area expanded by the bivector $b = x \wedge y$, namely, $\langle b, b \rangle = (\text{area})^2 \{x, y\} = \langle *L, *L \rangle \cong \rho^2$. This result is the analogue to that obtained in Section 7 of part I [1] where the area of the triangle expanded by the bivector was proportional to the value $(2j+1)$, j being the spin of the representation.

In order to construct the Lorentz invariant state sum we take a non-degenerate finite triangulation in a 4-dimensional simplices in such a way that all 3-dimensional and 2-dimensional subsimplices have space-like edge vectors which span space-like subspaces (Prop 7). We attach to each 2-dimensional face a simple irrep. of $\text{SO}(1,3,\mathbb{R})$ characterized by the parameter $[0, \rho]$. For these simple representations the intertwining condition is preserved (Prop 16 and remark 3).

The state sum is given by the expression

$$Z = \int_{\rho=0}^{\infty} d\rho \prod_{\text{triangle}} \rho^2 \prod_{\text{tetra}} \Theta_4(\rho'_1, \dots, \rho'_4) \prod_{\text{4-simplex}} I_{10}(\rho''_1, \dots, \rho''_{10}) \quad (65)$$

where ρ refers to all the faces in the triangulation, ρ' corresponds to the simple irreps attached to 4 triangles in the tetrahedra and ρ'' corresponds to the simple irrep attached to the 10 triangles in the 4-simplices.

The functions Θ_4 and I_{10} are defined as traces of recombination diagrams for the simple representations. The expressions eq. 65 are explicitly given as multiple integrals over the upper sheet H^3 of the 2-sheeted hyperboloid in Minkowski space. For the integrand we take

the zonal spherical functions eqs. 62, 63,

$$f^{[0,\rho]}(x,y) = \frac{1}{\rho} \frac{\sin(\rho \tau(x,y))}{\sinh(\tau(x,y))}, \quad f^{[0,\sigma]}(x,y) = \frac{1}{\sigma} \frac{\sinh(\sigma \tau(x,y))}{\sinh(\tau(x,y))} \quad (66)$$

for the chosen irrep of $SL(2, C)$ where $\tau(x,y)$ is the hyperbolic distance between x and y . If in remark 3 after Prop. 16 we follow [8] p. 3105 we can drop the second expression of eq. 66 for the supplementary series.

The contribution of a recombination diagram is given by a multiple integral of products of spherical functions.

For a tetrahedron we have

$$\Theta_4(\rho'_1, \dots, \rho'_4) = \frac{1}{2\pi^2} \int_H f^{\rho'_1}(x,y) \cdots f^{\rho'_4}(x,y) dy \quad (67)$$

where we have dropped one integral for the sake of normalization without loosing Lorentz symmetry.

For a 4-simplex we have

$$I_{10}(\rho_1, \dots, \rho_{10}) = \frac{1}{2\pi^2} \int_{H^4} \prod_{i < j}^5 f^{\rho_{ij}}(x_i, x_j) dx_1 dx_2 dx_3 dx_4 \quad (68)$$

Equations 66 to 68 define the state sum completely, that has been proved to be finite [9].

The asymptotic properties of the spherical functions are related to the Einstein-Hilbert action [1] giving a connection of the model with the theory of general relativity.

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